

# A necessary condition for the tightness of odd-dimensional combinatorial manifolds

Jonathan Spreer\*

## Abstract

*We present a necessary condition for  $(\ell - 1)$ -connected combinatorial  $(2\ell + 1)$ -manifolds to be tight. As a corollary, we show that there is no tight combinatorial three-manifold with Betti number at most two other than the boundary of the four-simplex and the nine-vertex triangulation of the three-dimensional Klein bottle.*

**MSC 2010:** 57Q15; 57N10; 05A19

**Keywords:** combinatorial manifolds, rsl-functions, slicings, tightness, tight triangulations,  $\sigma$ -vector

## 1 Introduction

Tight combinatorial manifolds are rare but very special objects. There are strong necessary conditions on when a combinatorial manifold can be tight and it is conjectured that all tight combinatorial manifolds are strongly minimal triangulations [13, Conjecture 1.3].

On the other hand, given a combinatorial manifold  $M$  it is difficult to check in general whether or not  $M$  is tight. One way to do this would be to look at all regular simplex-wise linear functions on  $M$  and check if they all have the minimum number of critical points, i.e., if they are all *perfect*, see [2] for an elaborate way to do this. As a consequence, necessary as well as sufficient conditions for tightness are highly sought after.

Here we establish new necessary conditions for the tightness of odd-dimensional combinatorial manifolds by analysing topological properties of slicings, i.e., co-dimension one normal submanifolds, which do not depend on the topology of the surrounding manifold.

As a result, we present upper bounds on the number of vertices of a combinatorial manifold  $M$  in terms of its Betti numbers, this way disqualifying large classes of topological manifolds from having tight triangulations at all.

In particular we prove the following result about  $(\ell - 1)$ -connected combinatorial  $(2\ell + 1)$ -manifolds complementing the results about  $(\ell - 1)$ -connected combinatorial  $2\ell$ -manifolds due to Kühnel [12].

**Theorem 1.1.** *Let  $\mathbb{M}$  be an  $\mathbb{F}$ -orientable compact closed  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifold admitting an  $n$ -vertex triangulation which is tight with respect to the field  $\mathbb{F}$ . Then*

$$\beta_\ell(M, \mathbb{F}) = \beta_{\ell+1}(M, \mathbb{F}) \geq \left[ (-1)^{\ell+1} \frac{(1 - \lfloor n/2 \rfloor)_{\ell+1} (1 - \lceil n/2 \rceil)_{\ell+1}}{(\ell + 1)! (1 - n)_{\ell+1}} \right] \quad (1.1)$$

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\*School of Mathematics and Physics, The University of Queensland, Brisbane QLD 4072, Australia. j.spreer@uq.edu.au

where  $(a)_n = a \cdot (a+1) \cdot (a+2) \cdot \dots \cdot (a+n-1)$  denotes the Pochhammer symbol.

As of today, the known cases of equality in (1.1) are the boundary of the simplex ( $\ell \geq 1$ ,  $\beta_\ell = 0$ ) and the 13-vertex triangulation of  $SU(3)/SO(3)$  ( $\ell = 2$  and  $\beta_\ell = 1$ ).

As a direct consequence any  $(\mathbb{F})$ -tight connected combinatorial three-manifold  $M$  with  $\beta_1(M, \mathbb{F}) \leq 2$  cannot have more than 12 vertices. Together with further results presented in Section 6 and extended computer experiments this leads to the following.

**Corollary 1.2.** *The boundary of the simplex and the nine-vertex three-dimensional Klein Bottle  $S^2 \times S^1$  are the only tight combinatorial three-manifolds with first Betti number at most two.*

## Acknowledgement

This work was supported by DIICCSRTE, Australia, under the Australia-India Strategic Research Fund (project AISRF06660). Furthermore, the author wants to thank Ole Warnaar for introducing him to hypergeometric sums and for the proof of Lemma 5.1.

## 2 Preliminaries

### 2.1 Combinatorial manifolds

A combinatorial  $d$ -manifold  $M$  is an abstract pure simplicial complex of dimension  $d$  such that all vertex links are triangulated standard  $PL$ -spheres. The  $f$ -vector of  $M$  is a  $(d+1)$ -tuple  $f(M) = (f_0, f_1, \dots, f_d)$  where  $f_i$  denotes the number of  $i$ -dimensional faces of  $M$ . The zero-dimensional faces of  $M$  are called *vertices*, the one-dimensional faces are called *edges* and the  $d$ -dimensional faces are referred to as *facets*. The set of vertices of  $M$  will be denoted by  $V(M)$  or just  $V$  if  $M$  is given by the context.

We call  $M$   $k$ -neighbourly, if  $f_{k-1} = \binom{f_0}{k}$ , i.e., if it contains all possible  $(k-1)$ -dimensional faces. An  $n$ -vertex combinatorial  $d$ -manifold  $M$  distinct from the boundary of the  $(d+1)$ -simplex can be at most  $(\lfloor \frac{d+2}{2} \rfloor)$ -neighbourly. In this case the  $f$ -vector of an odd-dimensional combinatorial manifold  $M$  is already determined to be the one of the boundary complex of the (even-dimensional) cyclic  $(d+1)$ -polytope with  $n$  vertices. This statement is known as the *Upper Bound Theorem* due to Novik [17] and Novik and Swartz [18].

Given a combinatorial manifold  $M$  with vertex set  $V(M)$  and  $W \subset V(M)$ , the simplicial complex

$$M[W] = \{\sigma \in M \mid V(\sigma) \subset W\},$$

i.e., the simplicial complex of all faces of  $M$  with vertex set in  $W$ , is called the *sub-complex of  $M$  induced by  $W$* .

### 2.2 Tightness

*Tightness* is a condition on subsets of Euclidean space generalising the notion of convexity: an object is tight if it is “as convex as possible”, i. e., as simple as possible, given its topological constraints. More precisely we have the following.

**Definition 2.1** (Tightness [12]). A compact connected subset  $M \subset E^d$  is called  $k$ -tight with respect to a field  $\mathbb{F}$  if for every open or closed half space  $h \subset E^d$  the induced homomorphism

$$H_k(h \cap M, \mathbb{F}) \rightarrow H_k(M, \mathbb{F})$$

is injective. If  $M \subset E^d$  is  $k$ -tight with respect to  $\mathbb{F}$  for all  $k$ ,  $0 \leq k \leq d$ , it is called *tight*.

Here and in the following  $H_\star$  denotes an appropriate homology theory (i.e., simplicial homology for our purposes). An  $n$ -vertex combinatorial manifold  $M$  with vertex set  $V = \{v_1, \dots, v_n\}$  is said to be *tight* if and only if its canonical embedding

$$i : M \rightarrow \mathbb{R}^n; \quad v_i \mapsto e_i$$

is tight with respect to at least one field  $\mathbb{F}$ .

Alternatively, there is the following combinatorial definition of tightness for combinatorial manifolds or even arbitrary abstract simplicial complexes.

**Definition 2.2** (Tightness [12, 2]). Let  $M$  be a combinatorial manifold with vertex set  $V(M)$  and let  $\mathbb{F}$  be a field. We say that  $M$  is tight with respect to  $\mathbb{F}$  if (i)  $M$  is connected, and (ii) for all subsets  $W \subset V(M)$  and for all  $0 \leq k \leq d$  the induced homomorphism

$$H_k(M[W], \mathbb{F}) \rightarrow H_k(M, \mathbb{F})$$

is injective.

### 2.3 Rsl-functions and slicings

Various discretisations of the concept of Morse theory provide important and powerful tools to investigate combinatorial manifolds. One of these is based on the following construction.

**Definition 2.3** (Rsl-function [12]). Let  $M$  be a combinatorial  $d$ -manifold. A function  $g : M \rightarrow \mathbb{R}$  is called *regular simplexwise linear (rsl)* if  $g(v) \neq g(w)$  for any two vertices  $w \neq v$  of  $M$ , and  $g$  is linear when restricted to a simplex of  $M$ .

A point  $x \in M$  is said to be *critical of index  $i$*  for an rsl-function  $g : M \rightarrow \mathbb{R}$  if

$$H_i(M_x, M_x \setminus \{x\}, \mathbb{F}) \neq 0$$

where  $M_x := \{y \in M \mid g(y) \leq g(x)\}$  and  $\mathbb{F}$  is a field. Furthermore, a critical point of index  $i$  is said to have *multiplicity  $m$*  if  $m = \text{rk } H_i(M_x, M_x \setminus \{x\}, \mathbb{F})$ . It follows that no point of  $M$  can be critical except possibly the vertices.

Using the notion of rsl-functions and critical vertices as defined above discrete analogues of the principal results of classical Morse can be obtained for combinatorial manifolds [6, 12]. In particular, for any field  $\mathbb{F}$  the sum of all critical points of an rsl-function  $g : M \rightarrow \mathbb{R}$  counted with multiplicity is greater or equal to the sum of all Betti numbers  $\beta_i(M, \mathbb{F})$ ,  $0 \leq i \leq d$ , and  $g$  is called *perfect* in the case of equality. Moreover, we have the following:

**Definition 2.4** (Slicing). Let  $M$  be a combinatorial  $d$ -manifold,  $g : M \rightarrow \mathbb{R}$  an rsl-function, and  $x \in \mathbb{R}$  such that  $x \notin f(v)$  for any vertex  $v \in V(M)$ . Then the pre-image  $S(g, x) = g^{-1}(x) \subset M$  is referred to as a *slicing* of  $M$ .

By construction, a slicing is a (polyhedral decomposition of a)  $(d-1)$ -manifold and for any ordered pair  $x < y$  the slicing  $g^{-1}(x)$  is isomorphic to  $g^{-1}(y)$  whenever  $g^{-1}([x, y])$  doesn't contain a vertex of  $M$ . In the following we will denote the class of isomorphic slicings between two adjacent vertices of  $g$  by  $S(g, v)$  where  $v$  is the vertex with the largest  $g$ -value smaller than  $x$ . It follows that any  $n$ -vertex combinatorial manifold has exactly  $2^{n-1} - 1$  such classes of slicings (not that the empty set is not counted as a slicing).

Let  $S(g, v) \subset M$  be a slicing of a combinatorial manifold  $M$ , then  $M \setminus S(g, v)$  splits into two connected components  $M^-$  and  $M^+$  where  $M^-$  denotes all points of  $x \in M$  with  $g(x) \leq g(v)$  and  $M^+$  denotes all  $y \in M$  such that  $g(y) > g(v)$ .

We can now use the theory of slicings and rsl-functions to give yet another definition of a tight combinatorial manifold.

**Definition 2.5** (Proposition 3.17 [12]). A combinatorial manifold  $M$  is tight with respect to some field  $\mathbb{F}$  if and only if all rsl-functions  $M \rightarrow \mathbb{R}$  are perfect with respect to  $\mathbb{F}$ .

## 2.4 Hypergeometric sums

Hypergeometric sums are a standard tool to prove combinatorial identities. Here, we will focus on their definition and some basic properties necessary to prove Theorem 1.1. For a more thorough introduction into the subject see [1, 3].

Given the *Pochhammer symbol*  $(a)_n = a(a+1)\dots(a+n-1)$  the generalised hypergeometric sum is defined as

$${}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; z \right) = \sum_{n \geq 0} \frac{(a_1)_n \dots (a_r)_n}{(b_1)_n \dots (b_s)_n} \frac{z^n}{n!},$$

and we have

$$\binom{a}{n} = (-1)^n \frac{(-a)_n}{n!}, \quad (2.1)$$

and

$$(a)_{2n} = 4^n (a/2)_n ((a+1)/2)_n. \quad (2.2)$$

Furthermore, for any positive integer  $n$  we have

$${}_2F_1 \left( \begin{matrix} a, -b \\ c \end{matrix} ; 1 \right) = \frac{(c-a)_b}{(c)_b}, \quad (2.3)$$

which is known as the *Chu-Vandermonde sum* [1, Corollary 2.2.3]; and

$${}_3F_2 \left( \begin{matrix} a, b, -c \\ d, 1+a+b-c-d \end{matrix} ; 1 \right) = \frac{(d-a)_c (d-b)_c}{(d)_c (d-a-b)_c}, \quad (2.4)$$

which is referred to as the *Pfaff-Saalschütz sum* [1, Theorem 2.2.6]. We will use these identities in Section 5 to prove Theorem 1.1.

## 3 Tightness and polyhedral critical point theory

Slicings and rsl-functions as explained in Section 2.3 are linked to the concept of tightness via Definition 2.5 and the following observation.

**Proposition 3.1.** *Let  $M$  be a combinatorial  $d$ -manifold which is tight with respect to a field  $\mathbb{F}$ , and let  $S(g, v) \subset M$  be a slicing. Then there exist an injective homomorphism*

$$H_i(S(g, v), \mathbb{F}) \rightarrow H_i(M, \mathbb{F}) \oplus H_{i+1}(M, \mathbb{F}).$$

*Proof.* Let  $S(g, v) \subset M$  be a slicing of a tight  $d$ -manifold  $M$  and let  $c_i \in H_i(S(g, v), \mathbb{F})$ . Then either  $c_i$  is an element of both  $H_i(M^+, \mathbb{F})$  and  $H_i(M^-, \mathbb{F})$ ; or  $c_i$  is the boundary of a sub-complex in both  $M^+$  and  $M^-$ , and both sub-complexes glued along their common boundary  $c_i$  become an element of  $H_{i+1}(M, \mathbb{F})$ .  $\square$

In particular it follows from Proposition 3.1 that  $\beta_i(S, \mathbb{F}) \leq \beta_i(M, \mathbb{F}) + \beta_{i+1}(M, \mathbb{F})$ . Even more, we can improve Proposition 3.1 in the following way.

**Proposition 3.2.** *Let  $M$  be a  $d$ -dimensional combinatorial manifold with set of vertices  $V$  which is tight with respect to  $\mathbb{F}$ . Then we have*

$$\max_g \sum_{i=0}^{d-1} \max_{1 \leq j \leq n} \beta_i(S(g, j), \mathbb{F}) \leq \sum_{i=0}^d \beta_i(M, \mathbb{F}).$$

*Proof.* First of all note that it is essential to first pick an rsl-function  $g$  and then look at the maximum Betti number of  $S(g, j)$  for all levels  $j$  of  $g$ . In particular the statement is false if we sum over the maximum Betti numbers of slicings associated to distinct rsl-functions.  $\square$

So far when talking about tightness we did not pay attention to the choice of field  $\mathbb{F}$ . However, if we choose  $\mathbb{F}$  carefully the problem of determining whether  $M$  is tight becomes more clear.

**Proposition 3.3** (Definition 2.7 [2]). *Let  $M$  be a combinatorial  $d$ -manifold. Then the following are equivalent:*

- (i)  $M$  is tight with respect to all fields,
- (ii)  $M$  is tight with respect to all fields of prime order,
- (iii)  $M$  is tight with respect to  $\mathbb{Q}$ .

Furthermore, we have

**Proposition 3.4** (Proposition 2.9(b) [2]). *Let  $M$  be a combinatorial  $d$ -manifold which is tight with respect to a field  $\mathbb{F}$ . Then  $M$  is also  $\mathbb{F}$ -orientable.*

Thus, (i) any non-orientable manifold is at most  $\mathbb{F}_2$ -tight and when talking about tightness we can always assume Poincaré duality holds, and (ii) any  $\mathbb{Q}$ -tight combinatorial manifold is also  $\mathbb{F}_2$ -tight. Furthermore, most combinatorial manifolds cannot be tight unless they are tight with respect to  $\mathbb{F}_2$  and there is no example of a tight combinatorial manifold known to the author for which this statement doesn't hold. For this reason we will treat  $\mathbb{F}_2$ -tightness as equivalent to tightness whenever possible and will only consider other fields when necessary.

In particular, whenever we talk about critical points and Betti numbers of a tight combinatorial manifold  $M$  we will omit the field  $\mathbb{F}$  and simply assume that either  $\mathbb{F} = \mathbb{F}_2$  or  $\mathbb{F}$  is chosen such that  $M$  is tight with respect to  $\mathbb{F}$ .

These simple observations allow us to impose bounds on the Betti numbers of tight combinatorial manifolds by looking at the topology of slicings. Note that in the case of odd-dimensional combinatorial manifolds this leaves us with studying even-dimensional slicings inside the manifold. This is of particular convenience because, as already pointed out in [12], questions about tightness are much harder to investigate in the odd-dimensional case due to the vanishing Euler characteristic. Looking at slicings enables us to use Euler characteristic arguments in the odd-dimensional case as well. This will complement observations made by Effenberger who gave a tightness criterion in the odd-dimensional case  $d \geq 5$  by looking at properties of the (even-dimensional) vertex links [6].

In Section 4 and 5 we present necessary conditions for tightness which are most powerful in the missing case of dimension three. However, let us first state a further *topological* constraint for tightness dimension three.

**Proposition 3.5.** *Let  $\mathbb{M}$  be an  $\mathbb{F}$ -orientable three-manifold of Heegaard genus  $\mathbf{g}$  such that  $\beta_1(\mathbb{M}, \mathbb{F}) < \mathbf{g}$ , then  $\mathbb{M}$  cannot have an  $\mathbb{F}$ -tight triangulation, that is, there is no  $\mathbb{F}$ -tight combinatorial manifold  $M \cong \mathbb{M}$ .*

*Proof.* Assume that  $M$  is a combinatorial manifold homeomorphic to  $\mathbb{M}$  with vertex set  $V(M) = \{v_1, \dots, v_n\}$  which is tight with respect to  $\mathbb{F}$ . Then every rsl-function  $g : M \rightarrow \mathbb{R}$  has exactly  $\beta_1(M, \mathbb{F})$  critical points of index one.

We will prove the statement by using  $g$  to construct a handlebody decomposition of  $M$  of genus  $\beta_1(M, \mathbb{F})$ . This will then contradict the assumption  $\beta_1(M, \mathbb{F}) < \mathbf{g}$ .

Without loss of generality, let  $g : M \rightarrow \mathbb{R}$  be given by  $g(v_1) < g(v_2) < \dots < g(v_n)$ . Start by setting  $H$  to be a neighbourhood around  $v_1$ , the unique critical point of index zero. Now proceed by considering every vertex in the ordering given by  $g$ . If  $v_i$  is not critical of index one, unite  $H$  with a neighbourhood of the edge going from  $v_{i-1}$  to  $v_i$  (note that  $M$  being tight implies that  $M$  is two-neighbourly and thus every two points are connected by an edge). If  $v_i$  is critical of index one and multiplicity  $m$ , the intersection of the link of  $v_i$  and  $H$  will have precisely  $m+1$  connected components, for each of them unite  $H$  with a small neighbourhood of an edge connecting it to  $v_i$ . As a result we will get a handle body  $H \subset M$  of genus  $\beta_1(M, \mathbb{F})$ .

To see that  $M \setminus H$  is also a handlebody consider the rsl-function  $-g : M \rightarrow \mathbb{R}$  restricted to  $M \setminus H$ . By construction  $-g$  will not have any critical points of index two or three in  $M \setminus H$  and only one critical point of index zero and  $\beta_1(M, \mathbb{F})$  critical points of index one. Thus,  $M \setminus H$  is a handle body and  $(H, M \setminus H)$  is a handle body decomposition of  $M$  of genus  $\beta_1(\mathbb{M}, \mathbb{F})$ . Contradiction to the assumption that  $M$  has Heegaard genus  $\beta_1(\mathbb{M}, \mathbb{F}) < \mathbf{g}$ . Hence,  $\mathbb{M}$  does not admit any  $\mathbb{F}$ -tight triangulation.  $\square$

The construction given above is a discrete version of a standard technique in smooth Morse theory. Given a  $d$ -manifold  $M$  and a Morse function  $f : M \rightarrow \mathbb{R}$ , a handle decomposition  $H$  (note that Heegaard decompositions are a special case of handle decompositions) can be constructed by adding an  $i$ -handle for each critical point of index  $i$  (see for instance [10, Section 4.2]). Thus, replacing the Heegaard genus by the size of a handle decomposition with the minimum number of handles might help to generalise this statement to the higher dimensional case.

## 4 An upper bound for tight odd-dimensional combinatorial manifolds

In this section we will look at the following situation. Let  $M$  be an  $n$ -vertex combinatorial  $(2\ell+1)$  manifold with vertex set  $V$  and  $f$ -vector  $f(M) = (n, f_1, \dots, f_{2k+1})$ . Furthermore, let

$$\mathbb{S}_k := \{S(g, v) \subset M \mid M^- \text{ contains } k \text{ vertices} \}$$

be the set of slicings of  $M$  separating  $k$  vertices from the other  $n-k$  vertices and let

$$\bar{f}(\mathbb{S}_k) = (\bar{f}_{0,k}, \bar{f}_{1,k}, \dots, \bar{f}_{2\ell,k})$$

be the *average  $f$ -vector* of  $S \in \mathbb{S}_k$ , that is, the sum over all  $f$ -vectors of slicings in  $\mathbb{S}_k$  divided by the cardinality of  $\mathbb{S}_k$ . With these definitions we can state

**Proposition 4.1.** *Let  $M$  be an  $n$ -vertex combinatorial  $(2\ell+1)$ -manifold with vertex set  $V$ . The average number of  $i$ -faces of a slicing  $S \in \mathbb{S}_k$  equals*

$$\bar{f}_{i,k} = f_{i+1} \left( 1 - \frac{\binom{k}{i+2} + \binom{n-k}{i+2}}{\binom{n}{i+2}} \right)$$

where  $f_{i+1}$  is the number of  $(i+1)$ -faces of  $M$ , and the average Euler characteristic of slicings in  $\mathbb{S}_k$  is given by

$$\bar{\chi}_k = \sum_{i=0}^{2\ell+1} (-1)^i \frac{f_i}{\binom{n}{i+1}} \left[ \binom{k}{i+1} + \binom{n-k}{i+1} \right].$$

In particular,  $\bar{\chi}_k$  does not depend on topological properties of  $M$ .

*Proof.* Fix a  $k$  vertex subset  $\Delta \subset V$  of  $M$  defining a slicing  $S \in \mathbb{S}_k$  and consider subsets  $\delta \subset V$  of size  $(i+2)$ .  $\delta$  intersects with  $S$  if and only if it is neither disjoint to nor contained in  $\Delta$ .

Now, the number of ways exactly  $j$  vertices,  $0 < j < i+2$ , of  $\delta$  lie in  $\Delta$  is  $\binom{k}{j}$  and for each of these choices, there are exactly  $\binom{n-k}{i+2-j}$  ways for the other  $i+2-j$  vertices of  $\delta$  to be chosen of the remaining  $n-k$  vertices. Summing up over  $j$  this leaves us with

$$\sum_{j=1}^{i+1} \binom{k}{j} \binom{n-k}{i+2-j}$$

possible intersections of subsets  $\delta$  with the slicing given by  $\Delta$ . Now, if  $\delta$  is an  $(i+1)$ -face of  $M$  each of these intersections results in exactly one face of  $S$ , and since there are  $\binom{n}{i+2}$  such subsets  $\delta$  but only  $f_{i+1}$   $(i+1)$ -faces we have for the average number of  $(i+1)$ -faces of  $S$

$$\bar{f}_{i,k} = \frac{f_{i+1}}{\binom{n}{i+2}} \sum_{j=1}^{i+1} \binom{k}{j} \binom{n-k}{i+2-j}. \quad (4.1)$$

Applying the Chu–Vandermonde sum (cf. Equation (2.3)) we get that

$$\sum_{j=1}^{i+1} \binom{k}{j} \binom{n-k}{i+2-j} = \binom{n}{i+2} - \left( \binom{k}{i+2} + \binom{n-k}{i+2} \right) \quad (4.2)$$

and the result follows by introducing the RHS of Equation (4.2) into Equation (4.1) for  $\bar{f}_{i,k}$ . Hence, we have

$$\begin{aligned} \bar{\chi}_k &= \sum_{i=0}^{2\ell} (-1)^i \bar{f}_{i,k} \\ &= \sum_{i=0}^{2\ell} (-1)^i f_{i+1} - \sum_{i=0}^{2\ell} (-1)^i \frac{f_{i+1}}{\binom{n}{i+2}} \left[ \binom{k}{i+2} + \binom{n-k}{i+2} \right] \\ &= n + \sum_{i=0}^{2\ell} (-1)^{i+1} \frac{f_{i+1}}{\binom{n}{i+2}} \left[ \binom{k}{i+2} + \binom{n-k}{i+2} \right] \\ &= \sum_{i=0}^{2\ell+1} (-1)^i \frac{f_i}{\binom{n}{i+1}} \left[ \binom{k}{i+1} + \binom{n-k}{i+1} \right] \end{aligned}$$

which proves the statement.  $\square$

Very recently, Swartz proved a similar but more general result grouping together discrete normal surfaces of a combinatorial manifold  $M$  where their corresponding dual one-cocycles lie in the same co-homology class [21]. Slicings are precisely those discrete normal surfaces where the dual one-cocycle is trivial in co-homology. Proposition 4.1 is thus a special case of Lemma 2.1 in [21].

The significance of Proposition 4.1 and the results in [21] is that they allow to compute the average Euler characteristic of slicings *independently* of the topology of the manifold  $M$  itself. The average Euler characteristic only depends on the number of faces of  $M$  in each dimension. A priori this information doesn't reveal any topological features of odd-dimensional combinatorial manifolds (e.g., by a theorem of Sarkaria and Walkup [19, 22], any three-manifold admits a triangulation with  $f$ -vector  $f = (n, \binom{n}{2}, 2(\binom{n}{2} - n), \binom{n}{2} - n)$  for  $n$  sufficiently large). However, by Proposition 4.1 almost all  $f$ -vectors of odd-dimensional combinatorial manifolds will give rise to either a strictly negative or a strictly positive average Euler characteristic. This postulates the existence of a slicing with non-trivial topology imposing lower bounds on the odd-dimensional Betti numbers of the surrounding manifold in the former, and lower bounds on the even-dimensional Betti numbers in the latter case. By Proposition 3.2 these bounds then directly translate to topological constraints for *tight* combinatorial manifolds.

In other words, Proposition 4.1 confirms the intuition one might have that a manifold where some of its topological features are not visible in homology has a small chance of admitting a tight triangulation at all (cf. Proposition 3.5).

In the next Section we will investigate how Proposition 4.1 can be restated in the important special case of tight  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifolds.

## 5 $(\ell - 1)$ -connected $(2\ell + 1)$ -manifolds

In [12] Kühnel gives a tightness criterion for  $(\ell - 1)$ -connected  $2\ell$ -manifolds. This result is complemented by a classification of tight combinatorial three-manifolds with first Betti number at most one and upper bounds on the number of vertices for simply connected five-manifolds with first Betti number at most one [12, Theorem 5.3 and Proposition 7.3]. Here, we want to generalise the latter set of results to the case of  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifolds (note that Kühnel's results for odd dimensions can be recovered from Theorem 1.1 as a special case).

Any tight  $(\ell - 1)$ -connected combinatorial  $(2\ell + 1)$ -manifold must be  $(\ell + 2)$ -neighbourly. To see this assume that there is a combinatorial  $(2\ell + 1)$ -manifold  $M$  with a minimal missing  $k$ -face  $\Delta$ , i.e., all proper subsets of  $\Delta$  span a face in  $M$  while  $\Delta$  is not a face of  $M$ ,  $k \leq \ell + 1$ . Then the slicing separating the span of  $\Delta$  and the rest of the manifold gives an obstruction to tightness. Now by the upper bound theorem [17, 18] we know that an  $(\ell + 2)$ -neighbourly combinatorial  $(2\ell + 1)$ -manifold  $M$  must have the  $f$ -vector of the boundary complex of the cyclic  $(2\ell + 2)$ -polytope and is thus determined by fixing the dimension and the number of vertices  $n$ .

Before we can prove Theorem 1.1, we first have to establish some useful tools.

**Lemma 5.1.** *Let*

$$s_{i,j}(k, n) = (-1)^i \binom{k}{i} \binom{n-1}{i}^{-1} \binom{n-j-1}{i-j} \binom{n-i}{2j-i}.$$

*Then for any  $j \geq 0$  we have*

$$\sum_{i \geq 0} s_{i,j}(k, n) = (-1)^j \binom{k}{j} \binom{n-k}{j} \binom{n-1}{j}^{-1} \quad (5.1)$$

*and for any integer  $i$*

$$\sum_{j \geq 0} s_{i,j}(k, n) = (-1)^i \binom{k}{i} \quad (5.2)$$

*holds.*



*Proof.* First note that both Equation (5.1) and (5.2) are finite sums since  $s_{ij}(k, n) = 0$  whenever  $j > i$  or  $i > 2j$ .

To show Equation 5.1 first perform an index shift followed by a re-arrangment of factorials resulting in

$$\begin{aligned} \sum_{i \geq 0} s_{i,j}(k, n) &\stackrel{i \mapsto i+j}{=} \sum_{i=0}^j s_{i+j,j} \\ &= (-1)^j \binom{k}{j} \binom{n-1}{j}^{-1} \binom{n-j}{j} {}_2F_1 \left( \begin{matrix} j-k, -j \\ j-n \end{matrix}; 1 \right). \end{aligned}$$

Now, by the Chu–Vandermonde sum (see Equation (2.3) or [1, Corollary 2.2.3])

$${}_2F_1 \left( \begin{matrix} a, -n \\ b \end{matrix}; 1 \right) = \frac{(b-a)_n}{(b)_n}$$

the statement follows. As for Equation (5.2) we have

$$\begin{aligned} \sum_{j \geq 0} s_{i,j}(k, n) &\stackrel{j \mapsto i-j}{=} \sum_{j=0}^{\lfloor i/2 \rfloor} s_{i,i-j}(k, n) \\ &= (-1)^i \binom{n-1}{i}^{-1} \binom{k}{i} \binom{n-i}{i} \sum_{j \in \mathbb{Z}} \frac{(n-i)_j (-i)_{2j}}{j! (n-2i+1)_{2j}} \\ &= (-1)^i \binom{n-1}{i}^{-1} \binom{k}{i} \binom{n-i}{i} \begin{cases} {}_3F_2 \left( \begin{matrix} n-i, -i/2, -(i-1)/2 \\ (1+n-2i)/2, (2+n-2i)/2 \end{matrix}; 1 \right) & \text{if } i \equiv 1(2) \\ {}_3F_2 \left( \begin{matrix} n-i, -(i-1)/2, -i/2 \\ (1+n-2i)/2, (2+n-2i)/2 \end{matrix}; 1 \right) & \text{otherwise} \end{cases} \end{aligned}$$

where the last step follows from Equation (2.2).

Hence, we have a hypergeometric sum satisfying the pre-conditions of the Pfaff–Saalschütz sum (see Equation (2.4) or [1, Theorem 2.2.6]) stating that

$${}_3F_2 \left( \begin{matrix} a, b, -n \\ d, 1+a+b-n-d \end{matrix}; 1 \right) = \frac{(d-a)_n (d-b)_n}{(d)_n (d-a-b)_n} \quad (5.3)$$

whenever  $n$  is a non-negative integer.

Now if  $i$  is odd we set  $n = (i-1)/2$ , and when  $i$  is even we set  $n = i/2$  and by re-arranging factorials we can see that most terms cancel out and Equation (5.2) follows.  $\square$

With these identities in mind, we can now prove the following.

**Lemma 5.2.** *The cyclic  $(2\ell+2)$ -polytope with  $n$  vertices  $C_{2m}(n)$  has*

$$f_{i-1}(C_{2m}(n)) = \frac{n}{n-i} \sum_{j=0}^{\ell+1} \binom{n-j-1}{i-j} \binom{n-i}{2j-i} \quad (5.4)$$

*faces of dimension  $(i-1)$ .*

*Proof.* If  $i > \ell + 1$  this follows directly from [4, Theorem 15.3.4].

Now let  $i \leq \ell + 1$ , that is, we have to show that

$$\frac{n}{n-i} \sum_{j=0}^{\ell+1} \binom{n-j-1}{i-j} \binom{n-i}{2j-i} = \binom{n}{i}.$$

First note that the summands in Equation (5.4) vanish unless  $0 \leq j \leq i \leq 2j$ . Thus, if  $i \leq \ell + 1$  we have

$$\sum_{j=0}^{\ell+1} \binom{n-j-1}{i-j} \binom{n-i}{2j-i} = \sum_{j \in \mathbb{Z}} \binom{n-j-1}{i-j} \binom{n-i}{2j-i}.$$

In addition, Lemma 5.1 states that

$$\sum_{j \in \mathbb{Z}} (-1)^i \binom{k}{i} \binom{n-1}{i}^{-1} \binom{n-j-1}{i-j} \binom{n-i}{2j-i} = (-1)^i \binom{k}{i}$$

which is equivalent to

$$\sum_{j \in \mathbb{Z}} \binom{n-j-1}{i-j} \binom{n-i}{2j-i} = \binom{n-1}{i} \quad (5.5)$$

and replacing the sum in Equation (5.4) with the RHS of Equation (5.5) we obtain the result.  $\square$

Now note that for  $\mathbb{F}$ -orientable  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifolds  $M$  we have  $\beta_1(M, \mathbb{F}) = \beta_2(M, \mathbb{F}) = \dots = \beta_{\ell-1}(M, \mathbb{F}) = 0$ . Furthermore, by Proposition 3.1 we know that  $\beta_\ell(S, \mathbb{F}) \leq \beta_\ell(M, \mathbb{F}) + \beta_{\ell+1}(M, \mathbb{F}) = 2\beta_\ell(M, \mathbb{F})$  and for any slicing  $S \subset M$  we have  $\chi(S) \geq 2 - \beta_\ell(S, \mathbb{F})$  if  $\ell$  is odd and  $\chi(S) \leq 2 + \beta_\ell(S, \mathbb{F})$  if  $\ell$  is even.

All together this results in

$$\beta_\ell(M, \mathbb{F}) = \beta_{\ell+1}(M, \mathbb{F}) \geq \begin{cases} \chi(S)/2 - 1 & \text{if } \ell \text{ is even} \\ 1 - \chi(S)/2 & \text{else} \end{cases}$$

and thus the inequality

$$\begin{aligned} \beta_\ell(M, \mathbb{F}) = \beta_{\ell+1}(M, \mathbb{F}) \geq (-1)^{\ell+1} (1 - \bar{\chi}_k/2) &= (-1)^{\ell+1} \left( 1 - \frac{1}{2} \left( \sum_{i=0}^{2\ell+1} (-1)^i \frac{f_i}{\binom{n}{i+1}} \left[ \binom{k}{i+1} + \binom{n-k}{i+1} \right] \right) \right) \\ &= \frac{(-1)^{\ell+1}}{2} \sum_{i=0}^{2\ell+2} \sum_{j=0}^{\ell+1} (-1)^i \frac{\binom{n-1-j}{i-j} \binom{n-i}{2j-i}}{\binom{n-1}{i}} \left[ \binom{k}{i} + \binom{n-k}{i} \right] \\ &= \frac{(-1)^{\ell+1}}{2} \sum_{i=0}^{2\ell+2} \sum_{j=0}^{\ell+1} [s_{i,j}(k, n) + s_{i,j}(n-k, n)], \end{aligned}$$

with  $s(k, n)$  as defined in Lemma 5.1, must hold for all  $k \leq n/2$ .

*Proof of Theorem 1.1.* By the above calculations and Lemma 5.2 it suffices to show that

$$\sum_{i=0}^{2\ell+2} \sum_{j=0}^{\ell+1} s_{i,j}(k, n) = \frac{(1-k)_{\ell+1} (1-n+k)_{\ell+1}}{(\ell+1)! (1-n)_{\ell+1}}$$

for all  $k \leq n/2$ . Applying Equation (5.1) from Lemma 5.1 and letting  $c$  tend to  $-n$  in the Pfaff-Saalschütz sum Equation (5.3) we have

$$\begin{aligned} \sum_{i=0}^{2\ell+2} \sum_{j=0}^{\ell+1} s_{i,j}(k, n) &= \sum_{j \geq 0} \sum_{i \geq 0} s_{i,j}(k, n) \\ &= \sum_{j \geq 0} (-1)^j \binom{k}{j} \binom{n-k}{j} \binom{n-1}{j}^{-1} \\ &= \frac{(1-k)_{\ell+1} (1-n+k)_{\ell+1}}{(\ell+1)! (1-n)_{\ell+1}}. \end{aligned}$$

Note that the above statement is most restrictive in the case  $k = \lfloor n/2 \rfloor$ . Finally, multiplying the equation by  $-1$  whenever  $\ell$  is even completes the proof.  $\square$

In dimensions  $d = 2\ell + 1$ ,  $1 \leq \ell \leq 15$ , Theorem 1.1 translates to the following upper bounds for vertex numbers of tight  $(\ell - 1)$ -connected  $d$ -manifolds.

$\beta_\ell(M) \backslash d$	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
0	<b>5</b>	<b>7</b>	<b>9</b>	<b>11</b>	<b>13</b>	<b>15</b>	<b>17</b>	<b>19</b>	<b>21</b>	<b>23</b>	<b>25</b>	<b>27</b>	<b>29</b>	<b>31</b>	<b>33</b>
1	10	<b>13</b>	16	19	22	25	28	31	34	37	40	43	46	49	52
2	12	15	17	20	23	26	29	32	35	39	41	45	48	51	54
3	14	16	19	21	24	27	30	33	36	—	42	—	—	—	—
4	15	17	—	22	25	28	31	34	37	40	43	46	49	52	55
5	17	18	20	23	26	29	—	—	—	—	—	—	—	—	—
6	18	19	21	—	—	—	32	35	38	41	44	47	50	53	56
7	19	—	—	24	27	—	—	—	—	—	—	—	—	—	—
8	20	20	22	—	—	30	33	—	—	—	—	—	—	—	—
9	21	21	—	25	—	—	—	36	39	42	—	—	—	—	—
10	22	—	23	—	28	—	—	—	—	—	45	48	51	54	57

— denotes the same value as above.

Bold entries denote cases where tight triangulations exist.

In particular, the bound of Theorem 1.1 is attained for  $d = 5$  and  $\beta_2(M) = 1$  as there is a tight 13-vertex triangulation of  $SU(3)/SO(3)$  described in [13].

*Remark 5.3.*  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifolds are less restrictive than an  $(\ell - 1)$ -connected  $2\ell$ -manifold. This is most apparent in the case of  $\ell = 1$  where Theorem 1.1 holds for all connected three-manifolds (cf. Section 6 where we will deal with the three-dimensional case in more detail). This is of particular interest as dimension three is the unique non-trivial odd dimension where Effenberger's tightness criterion [6] cannot be applied.

Following Theorem 1.1 a tight  $n$ -vertex  $(\ell - 1)$  connected  $(2\ell + 1)$ -manifold  $M$  must satisfy

$$\beta_\ell(M, \mathbb{F}) \geq \frac{n^{\ell+1}}{4^{\ell+1}(\ell+1)!} + o(n^{\ell+1}). \quad (5.6)$$

For three-manifolds we thus have  $\beta_1(M, \mathbb{F}) \geq \frac{n^2}{32} + o(n^2)$ . On the other hand it follows from [18] and [16, Theorem 5] that  $\beta_1(M, \mathbb{F}) \leq \frac{n^2}{20} + o(n^2)$  for all combinatorial three-manifolds. Thus, as  $n$  tends

to infinity, the number of vertices of a tight combinatorial three-manifold  $M$  cannot be greater than  $\sqrt{1.6} \simeq 1.265$  times the minimum number of vertices needed to triangulate any manifold  $N$  with first Betti number  $\beta_1(M, \mathbb{F})$ . In particular, the conjecture by Lutz and Kühnel stating that tight combinatorial manifolds are strongly minimal [13, Conjecture 1.3] cannot be far from being true in dimension three.

*Remark 5.4* (Comparison with Bagchi and Datta's tightness criterion). In [2] the authors give a tightness criterion for all combinatorial manifolds in Walkup's class  $\mathcal{W}_k^*(d)$ . In the  $(\ell - 1)$ -dimensional  $(2\ell + 1)$ -dimensional case these  $n$ -vertex combinatorial manifolds must satisfy

$$\beta_\ell(M, \mathbb{F}) = \frac{\binom{n-\ell-3}{\ell+1}}{\binom{2\ell+3}{\ell+1}} \simeq \frac{(\ell+2)!}{(2\ell+3)!} n^{\ell+1} + o(n^\ell).$$

Comparing this to Equation (5.6) we get that, for  $n$  tending to infinity, any tight combinatorial  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifold cannot have more than

$$\sqrt[\ell+1]{\frac{4^{\ell+1}(\ell+1)!(\ell+2)!}{(2\ell+3)!}} = 4 \cdot \exp^{\frac{1}{\ell+1}[\ln(\ell+1)! + \ln(\ell+2)! - \ln(2\ell+3)!]}$$

times the vertices than any tight combinatorial  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifold in  $\mathcal{W}_k^*(d)$ . And since

$$\begin{aligned} \ln(\ell+1)! + \ln(\ell+2)! - \ln(2\ell+3)! &= \sum_{x=1}^{\ell+1} \ln(x) + \sum_{x=1}^{\ell+2} \ln(x) - \sum_{x=1}^{2\ell+3} \ln(x) \\ &= \sum_{x=1}^{\ell+2} \ln(x) - \sum_{x=1}^{\ell+1} \ln(x+1/2) - (\ell+1)\ln(4) \\ &< \ln(\ell+2) - (\ell+1)\ln(4), \end{aligned}$$

we get

$$\sqrt[\ell+1]{\frac{4^{\ell+1}(\ell+1)!(\ell+2)!}{(2\ell+3)!}} \xrightarrow{\ell \rightarrow \infty} 1.$$

Hence, Theorem 1.1 gives a necessary condition for the tightness of arbitrary combinatorial  $(\ell - 1)$ -connected  $(2\ell + 1)$ -manifolds which asymptotically (in  $n$  then  $\ell$ ) tends to Bagchi and Datta's tightness criterion for triangulations in  $\mathcal{W}_k^*(d)$ .

## 6 Three-manifolds with small first Betti number

In this section we will focus on combinatorial three-manifolds with low first Betti numbers, proving that the only tight combinatorial three-manifolds with first Betti number at most two are the five-vertex boundary of the four-simplex and the nine-vertex triangulation of the three-dimensional Klein bottle  $S^2 \times S^1$  [12].

In [2] a tightness criterion for combinatorial  $d$ -manifolds  $M$  is given based on calculations in the vertex links of  $M$ . More precisely, a condition on the  $\sigma$ -vectors of the vertex links is given for  $M$  to be tight. These are defined as follows.

**Definition 6.1** ( $\sigma$ -vector). Let  $M$  be a combinatorial  $d$ -manifold with vertex set  $V(M) = \{v_1, \dots, v_n\}$ . The  $\sigma$ -vector  $\sigma(M) = (\sigma_0, \dots, \sigma_d)$  of  $M$  is defined by

$$\sigma_i = \sum_{j=0}^n \frac{1}{\binom{n}{j}} \sum_{\substack{W \subset V(M), \\ |W|=j}} \tilde{\beta}_i(M[W]), \quad 0 \leq i \leq d.$$

For  $d = 3$ , tight combinatorial manifolds must be two-neighbourly and any two-neighbourly combinatorial three-manifold is tight if and only if it is one-tight [12, Proposition 3.18], [2, Proposition 2.9(b)]. Thus, in dimension three we have the following tightness criterion.

**Proposition 6.2** (Corollary of Theorem 2.10 [2]). *Let  $M$  be a combinatorial three-manifold with vertex set  $V$ . Then  $M$  is  $\mathbb{F}$ -tight if and only if  $M$  is two-neighbourly and*

$$\frac{1}{|V|} \sum_{v \in V} \sigma_0(\text{lk}_M(v)) = \beta_1(M, \mathbb{F}) - 1, \quad (6.1)$$

that is, the average value for  $\sigma_0$  over all vertex links must equal the first Betti number minus one.

However, not all combinations of vertex links satisfying Equation (6.1) have a chance of being the set of links of a tight combinatorial three-manifold. Thus, we need to take a closer look at some essential properties of vertex links of tight combinatorial three-manifolds.

**Lemma 6.3** (Property  $T_k$ ). *Let  $S$  be an  $n$ -vertex triangulation of a two-sphere with one-skeleton  $G = \text{skel}_1(S)$  which occurs as a link of a tight combinatorial three-manifold  $M$  with  $k = \beta_1(M, \mathbb{F})$ , then the following conditions must be satisfied for at least one fixed field  $\mathbb{F}$ .*

- (i)  $k$  satisfies the condition given by Theorem 1.1 for  $n + 1$  vertices,
- (ii)  $G$  does not have an independent set of size  $k + 2$ , and
- (iii)  $G$  does not have an induced subgraph with six vertices and  $k + 1$  connected components.

If  $S$  satisfies all of the above properties it is said to have Property  $T_k$ .

Note that (ii) and (iii) are independent of the choice of  $\mathbb{F}$ .

*Proof.* For (i) note that the two-neighbourliness condition implies that any tight combinatorial manifold  $M$  with  $n$ -vertex vertex links has  $n + 1$  vertices. Then apply Theorem 1.1.

For (ii) note that any independent set of size  $k + 2$  in the one-skeleton  $G$  of a vertex link of some vertex  $v \in V(M)$  gives rise to an rsl-function with a critical point of index one and multiplicity  $k + 1$ : define  $g : M \rightarrow \mathbb{R}$  such that  $g(w) < g(v)$  for all vertices  $w$  in the independent set and  $g(v) < g(u)$  for all other vertices. Now  $g$  clearly has more than  $k$  critical points of index one counted by multiplicity. Hence,  $g$  is not perfect and  $M$  cannot be tight.

For (iii) let  $G$  be the 1-skeleton of the vertex link of  $v \in V(M)$ , and let  $W \subset V(M) \setminus \{v\}$ ,  $W = \{w_1, w_2, \dots, w_6\}$ , such that  $G[W]$  has  $k + 1$  connected components. Now let  $G'$  be the 1-skeleton of the vertex link of  $w_1$  in  $M$ . Since  $G'$  is planar, it does not contain a complete graph with five vertices and hence there must be a missing edge in the induced subgraph  $G'[W \setminus \{w_1\}]$ . Without loss of generality, let  $\{w_2, w_3\}$  be that missing edge.

With this setup in mind choose an rsl-function of  $g : M \rightarrow \mathbb{R}$  such that

$$g(w_2) < g(w_3) < g(w_1) < g(w_4), g(w_5), g(w_6) < g(v) < \dots$$

It follows from the construction that  $w_1$  is critical of index one and multiplicity one and  $v$  is critical of index one and multiplicity  $k$  and  $M$  cannot be tight.  $\square$

Now in the three-dimensional case Theorem 1.1 and [16, Theorem 5] give us the following upper and lower bounds on the vertex numbers of tight combinatorial three-manifolds with prescribed first Betti number.

$\beta_1(M, \mathbb{F})$	lower b.	upper b.
0	<b>5</b>	<b>5</b>
1	<b>9</b>	10
2	11	12
3	13	14
4	14	15
5	15	17
6	16	18
7	17	19
8	18	20
9	18	21
10	19	22
11	20	23
12	20	24

Bold entries denote cases where tight triangulations exist.

In the following we will use these bounds together with Property  $T_k$ ,  $k \leq 2$ , the classification of sphere triangulations up to 11 vertices (triangulations are taken from [14]), as well as the classification of three-manifold triangulations up to 11 vertices [20] to give an alternative classification of tight combinatorial three-manifolds with first Betti number at most one. Furthermore, we will show that there are no tight combinatorial three-manifolds  $M$  with  $\beta_1(M, \mathbb{F}) = 2$  for all fields  $\mathbb{F}$ .

**The case  $\beta_1(M, \mathbb{F}) = 0$**

A tight combinatorial three-manifold with vanishing first Betti number must be three-neighbourly (otherwise consider an rsl-function separating a minimal missing triangle from the rest of the triangulation). It follows immediately that the only tight combinatorial homology three-sphere is the boundary of the simplex.

**The case  $\beta_1(M, \mathbb{F}) = 1$**

The case  $\beta_1(M, \mathbb{F}) = 1$  is due to Kühnel [12, Theorem 5.3]. Alternatively, it can also be followed from Theorem 1.1 and the classification of combinatorial three-manifolds up to 11 vertices [20]. Here, we will give yet another proof using Theorem 1.1 and the (much smaller) classification of two-sphere triangulations up to nine vertices.

Recall that in a tight  $n$ -vertex combinatorial three-manifold all vertex links have to be  $(n - 1)$ -vertex two-sphere triangulations. Now, following Theorem 1.1, a tight combinatorial three-manifold with  $\beta_1(M, \mathbb{F}) = 1$  needs to have either nine or ten vertices.

**Case  $n = 9$ :** We have 14 triangulations of the two-sphere with eight vertices with the following  $\sigma_0$ -values.

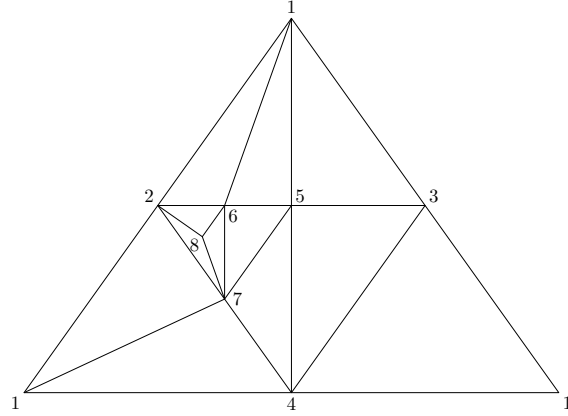


Figure 6.1: The unique eight-vertex two-sphere triangulation with  $\sigma_0 = 0$  and property  $T_1$ .

$\sigma_0$	$\approx \sigma$	# two-sphere triangulations
$-2/7$	$-0.2857$	1
$-8/35$	$-0.2285$	1
$-27/140$	$-0.1928$	1
$-9/70$	$-0.1285$	4
0	0	7
Total:		14

Hence, to satisfy Equation (6.1) only the seven triangulations with  $\sigma_0$ -value equal to zero can be considered (these are precisely the seven stacked eight-vertex two-spheres, see [5] for a more general observation on the  $\sigma_0$ -value of two-sphere triangulations). Amongst these seven triangulations only the triangulation presented in Figure 6.1 satisfies Property  $T_1$  and thus any tight combinatorial three-manifold  $M$  with  $\beta_1(M, \mathbb{F}) = 1$  must have nine isomorphic vertex links of that type. By virtue of the enumeration algorithm presented below, this leads to the unique nine-vertex triangulation of the three-dimensional Klein bottle.

**Case  $n = 10$ :** We need to consider the classification of nine-vertex two-sphere triangulations for which we have the following  $\sigma_0$ -values.

$\sigma_0$	$\approx \sigma_0$	# two-sphere triangulations
$1/21$	$0.0476$	1
$2/21$	$0.0952$	1
$8/63$	$0.1269$	1
$23/126$	$0.1825$	2
$3/14$	$0.2142$	1
$2/9$	$0.2222$	1
$31/126$	$0.246$	1
$2/7$	$0.2857$	7
$5/14$	$0.3571$	11
$1/2$	$0.5$	24
Total:		50

In particular the smallest  $\sigma_0$ -value is greater than zero and thus Equation (6.1) can never hold. Hence no tight ten-vertex combinatorial three-manifold  $M$  with  $\beta_1(M, \mathbb{F}) = 1$  exist.

**The case  $\beta_1(M, \mathbb{F}) = 2$**

Again, we have to consider two cases.

**Case  $n = 11$ :** The classification of 11-vertex combinatorial three-manifolds tells us that there is no combinatorial three-manifold with  $\leq 11$  vertices and first Betti number greater than one.

**Case  $n = 12$ :** There are 1249 triangulations of the sphere with 11-vertices. A computer search testing all 1249 triangulations of the two-sphere with 11 vertices resulted in 22 triangulations with property  $T_2$  having 18 distinct  $\sigma_0$  values. These 18  $\sigma_0$ -values allow 29 combinations of size 12 with an average of  $\beta_1(M, \mathbb{F}) - 1 = 1$ , resulting in 50 combinations of vertex links with this property. Amongst these 50 combinations of vertex links, 42 have at least one vertex degree occurring an odd number of times. Such a combination cannot be the set of vertex links of a closed combinatorial manifold since pairs of vertex stars in the vertex links have to meet in an edge link of the surrounding manifold and hence the number of vertices of a given degree over all vertex links of a triangulation must always be an even number. This leaves us with eight potential combinations of vertex links consisting of 11 distinct sphere triangulations (see table below).

Isomorphism signature**	$\sigma_0$	degree sequence*
cdef.e.gbhag.haibibjbkbkbjbkbkhhk	2254/1155	$4^2 5^8 6^1$
cdef.e.gbhag.haibibjbkbkbjbkbkfjdk	2296/1155	$4^3 5^6 6^2$
cdef.e.fgbgahbibi.hajcjdcbkbbkbk	2323/1155	$4^3 5^6 6^2$
cdef.e.fgbgahbibi.bjbkbj.iakbkeiej	2367/1155	$4^4 5^4 6^3$
cdef.e.fgbgahbibi.hajckbkakbkbk	2370/1155	$4^4 5^5 6^1 7^1$
cdef.e.fgbgahbibi.bjciajbkbkbkbk	2416/1155	$4^4 5^4 6^3$
cdef.e.fgbgahbibi.bjckaj.kajejbk	2416/1155	$4^4 5^5 6^1 7^1$
cddeafbgaf.haibi.hajbjbjkbkkek	2422/1155	$3^1 4^1 5^7 6^2$
cdef.e.fgbgahbibi.bjbkbjckakek	2448/1155	$4^4 5^5 6^1 7^1$
cddeafbgaf.haibi.hajbjkbkbkbkbk	2454/1155	$3^1 4^2 5^5 6^3$
cddeafbgaf.gahhbibi.bjbkbjbkbkek	2564/1155	$3^1 4^2 5^5 6^3$

\* The degree sequence  $d_1^{e_1} d_2^{e_2} \dots d_m^{e_m}$  of a two-sphere triangulation  $S$  denotes that  $S$  has  $e_i$  vertices of degree  $d_i$ ,  $1 \leq i \leq m$ .

\*\* The isomorphism signature of a combinatorial manifold uniquely determines its isomorphism type, i.e., two combinatorial manifolds have equal isomorphism signature if and only if they are isomorphic. The isomorphism signature given in this table coincides with the one used by simpcomp [7, 8, 9]. Use the function `SCFromIsoSig(...)` to generate the complexes. See the manual for details.

For the remaining eight combinations of twelve 11-vertex two-sphere triangulations we apply an exhaustive search for tight combinatorial three-manifolds having any of these combinations as their vertex links. The search essentially fixes an ordering of the vertex links using the fact that by the two-neighbourliness the intersection of any two pairs of vertex stars contains an edge star of the three-manifold. Then it starts combining vertex stars in this ordering looping over all matching pairs of vertices in two consecutive links (i.e., vertices of equal degree in the two-sphere triangulation), all rotations and reflections of a matching pair and all permutations of the remaining vertices. In each step the complex is tested if (i) it is the sub-complex of a two-neighbourly three-manifold and (ii) if it satisfies a generalised version of property  $T_k$ . Whenever a complex is valid an additional link is added to the existing combination. If it fails one of the



tests the link added last is removed and the next option is tried. The algorithm in full detail is available from the author upon request.

The above search yielded zero tight combinatorial 12-vertex three-manifolds and hence we have the following.

**Theorem 6.4.** *There is no tight 12-vertex combinatorial three-manifold.*

Corollary 1.2 follows instantly from the above observations combined with Theorem 1.1.

**The case  $\beta_1(M, \mathbb{F}) = 3$**

For  $\beta_1(M, \mathbb{F}) = 3$  any tight combinatorial three-manifold has to have either 13 or 14 vertices. For 14 vertices an analysis of the  $\sigma_0$  values of all 13-vertex triangulations of the two-sphere results in a minimal  $\sigma_0$ -value of  $26971/12870 \sim 2.09565 > \beta_1(M, \mathbb{F}) - 1$  and hence no tight 14-vertex combinatorial three-manifold with first Betti number three can exist.

The 13-vertex case has to be left open at this point. However, it follows from the above observation together with the Lutz-Kühnel conjecture [13, Conjecture 1.3] and a statement conjectured in [15, Conjecture 32].

**Corollary 6.5.** *The only tight combinatorial three-manifolds with  $\beta_1(M, \mathbb{F}) \leq 3$  for any field  $\mathbb{F}$  are the boundary of the four-simplex and the nine-vertex triangulation of  $S^1 \times S^2$ .*

*Proof.* By the above, a tight combinatorial three-manifold  $M$  with  $\beta_1(M, \mathbb{F}) = 3$  must have 13 vertices. The most natural candidates for the topological type of such a tight combinatorial three-manifold are  $(S^2 \times S^1)^{\#3}$  and  $(S^2 \times S^1)^{\#3}$ . However, these manifolds admit non-neighbourly 13-vertex triangulations [16, Corollary 32] and thus a tight triangulation of these manifolds cannot be strongly minimal. The statement now follows as a consequence of the Lutz-Kühnel conjecture [13, Conjecture 1.3] and [15, Conjecture 35] where it is conjectured that the only manifolds with  $\beta_1(M, \mathbb{F}) = 3$  admitting a 13-vertex triangulation are  $(S^2 \times S^1)^{\#3}$  and  $(S^2 \times S^1)^{\#3}$ .  $\square$

In particular, Corollary 6.5 implies that modulo [13, Conjecture 1.3] and [15, Conjecture 32] there is no tight triangulation of the three-torus.

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